

QUESTION 1

The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

[10 pts] a) Evaluate the integral $\int \frac{dx}{(x^2 - 1)^{3/2}}$ by using trigonometric substitution.[10 pts] b) Evaluate the integral $\int \frac{x^2 + 5x + 2}{(x + 1)(x^2 + 1)} dx$.[5 pts] c) Find $\frac{d}{dx} \int_{\sqrt{x}}^0 \sin(t^2) dt$.

a) $x = \sec t \quad dx = \sec t \cdot \tan t \, dt \Rightarrow (x^2 - 1)^{3/2} = (\sec^2 t - 1)^{3/2} = (\tan^2 t)^{3/2}$

$$\int \frac{dx}{(x^2 - 1)^{3/2}} = \int \frac{\sec t \cdot \tan t \, dt}{\tan^3 t} = \int \frac{1}{\cos t} \cdot \frac{\cos^2 t}{\sin^2 t} dt = \int \frac{\cos t}{\sin^2 t} dt$$

$$\left. \begin{array}{l} u = \sin t \\ du = \cos t \, dt \end{array} \right\} \Rightarrow \int \frac{dx}{(x^2 - 1)^{3/2}} = \int \frac{du}{u^2} = -\frac{1}{u} + C' = -\frac{1}{\sin t} + C = -\frac{x}{\sqrt{x^2 - 1}} + C$$

b) $\frac{x^2 + 5x + 2}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$

$$\Rightarrow A(x^2 + 1) + (Bx + C)(x + 1) = x^2 + 5x + 2 \Rightarrow Ax^2 + A + Bx^2 + Bx + Cx + C = x^2 + 5x + 2$$

$$\Rightarrow (A + B)x^2 + (B + C)x + A + C = x^2 + 5x + 2$$

$$\Rightarrow \left. \begin{array}{l} A + B = 1 \\ B + C = 5 \\ A + C = 2 \end{array} \right\} \Rightarrow 2A + 2B + 2C = 8 \Rightarrow A + B + C = 4 \Rightarrow A = -1, B = 2, C = 3$$

$$\frac{x^2 + 5x + 2}{(x + 1)(x^2 + 1)} = \frac{-1}{x + 1} + \frac{2x + 3}{x^2 + 1}$$

$$\int \frac{x^2 + 5x + 2}{(x + 1)(x^2 + 1)} dx = \int \frac{-dx}{x + 1} + \int \frac{2x + 3}{x^2 + 1} dx = -\int \frac{dx}{x + 1} + \int \frac{2x dx}{x^2 + 1} + 3 \int \frac{dx}{x^2 + 1}$$

$$= -\ln|x + 1| + \ln(x^2 + 1) + 3 \tan^{-1} x + C$$

c) $\frac{d}{dx} \int_{\sqrt{x}}^0 \sin(t^2) dt = \frac{d}{dx} \left(- \int_0^{\sqrt{x}} \sin(t^2) dt \right) \stackrel{\text{By the FTC}}{=} -\sin(\sqrt{x})^2 \cdot \frac{1}{2\sqrt{x}} = -\frac{\sin x}{2\sqrt{x}}$

QUESTION 2

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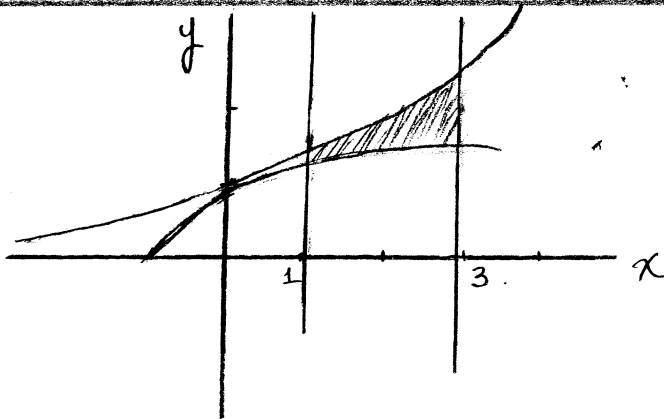
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[7 pts] a) Find the area of the region **R** in the first quadrant enclosed above by $y = e^x$ and below by $y = \sqrt{x+1}$ where $1 \leq x \leq 3$ using definite integral.

[13 pts] b) Find the area of the region **R** inside the circle $r = 3 \cos \theta$ and outside the cardioid $r = 1 + \cos \theta$.

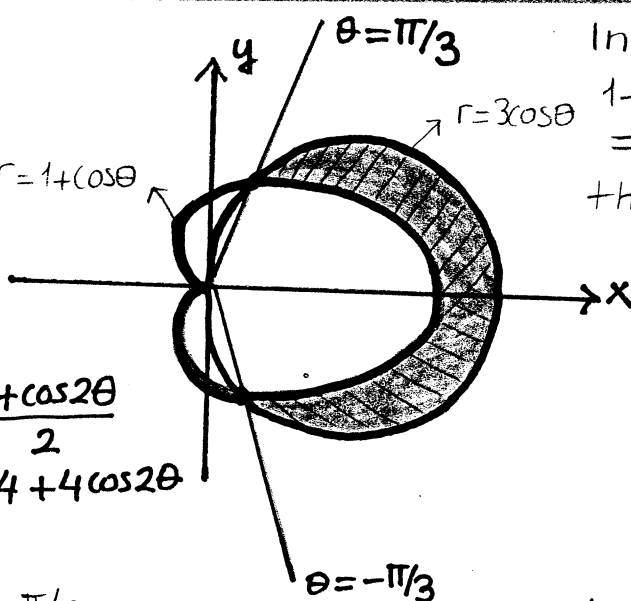
a)



$$A = \int_1^3 (e^x - \sqrt{x+1}) dx$$

$$A = \left[e^x - \frac{2}{3} (x+1)^{3/2} \right] \Big|_1^3 = e^3 - \frac{2}{3} 4^{3/2} - e + \frac{2}{3} 2^{3/2} = e^3 - e + \frac{4\sqrt{2}-16}{3}$$

b)



Intersection points:

$$1 + \cos \theta = 3 \cos \theta \Rightarrow 2 \cos \theta = 1 \Rightarrow \cos \theta = 1/2 \Rightarrow \theta = \pm \pi/3 \text{ (The graph also gives the point of intersection } (0,0))$$

$$A = 2 \int_0^{\pi/3} \frac{1}{2} [(3 \cos \theta)^2 - (1 + \cos \theta)^2] d\theta$$

$$A = \int_0^{\pi/3} (9 \cos^2 \theta - 1 - 2 \cos \theta - \cos^2 \theta) d\theta$$

$$A = \int_0^{\pi/3} (8 \cos^2 \theta - 2 \cos \theta - 1) d\theta = \int_0^{\pi/3} (4 \cos 2\theta - 2 \cos \theta + 3) d\theta$$

$$A = (2 \sin 2\theta - 2 \sin \theta + 3\theta) \Big|_0^{\pi/3} = 2 \sin \frac{2\pi}{3} - 2 \sin \frac{\pi}{3} + 3 \cdot \frac{\pi}{3} = \pi //$$

QUESTION 3

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[10 pts] a) Find the length of the curve $y = \ln(\cos x)$ where $0 \leq x \leq \frac{\pi}{3}$.[15 pts] b) Let R be the region bounded above by the curve $y = \cosh x$, below by the curve $y = \sinh x$, from the left by the y -axis and from the right by the line $x = 1$.

- Draw the region R . (2 pts)
- Find the volume of the solid generated by revolving the region R about the x -axis using Disk/Washer method. (8 pts)
- Write the definite integral that calculates the volume of the solid generated by revolving the region R about y -axis using Shell method. Do not evaluate the integral. (5 pts)

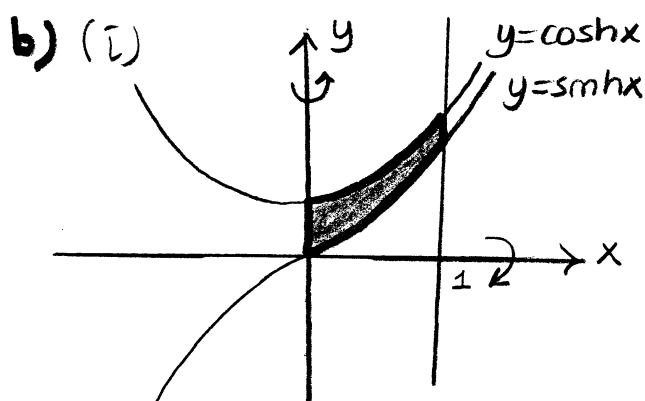
a) $y = \ln(\cos x)$ $\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$

$$L = \int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^{\pi/3} \sqrt{1 + \tan^2 x} dx = \int_0^{\pi/3} |\sec x| dx = \int_0^{\pi/3} \sec x dx$$

$$= \int_0^{\pi/3} \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int_0^{\pi/3} \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} dx$$

$$= \ln |\sec x + \tan x| \Big|_0^{\pi/3} = \ln |\sec \pi/3 + \tan \pi/3| - \ln |\sec 0 + \tan 0|$$

$$= \ln(2 + \sqrt{3}) - \ln 1 = \ln(2 + \sqrt{3})$$



(ii) $V_{y=0} = \pi \int_0^1 (\cosh^2 x - \sinh^2 x) dx$

$$= \pi \int_0^1 1 dx = \pi$$

(iii) $V_{x=0} = 2\pi \int \frac{x}{r} (\cosh x - \sinh x) dx$

$$\cosh^2 x - \sinh^2 x = 1$$

QUESTION 4

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[12 pts] a) Investigate the convergence or divergence of the improper integral $\int_0^1 x \ln x dx$ by evaluating the integral.

[8 pts] b) Investigate the convergence or divergence of the improper integral $\int_3^\infty \frac{1}{x - e^{-x}} dx$ by using a convenient test.

[10 pts] c) Find the limit $\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$.

$$a) \int x \ln x dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{dx}{x} = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

$$\left\{ \begin{array}{l} u = \ln x \quad du = \frac{dx}{x} \\ dv = x dx \quad v = \frac{x^2}{2} \end{array} \right\}$$

$$\begin{aligned} \int_0^1 x \ln x dx &= \lim_{a \rightarrow 0^+} \int_a^1 x \ln x dx = \lim_{a \rightarrow 0^+} \frac{1}{4} \left(2x^2 \ln x - x^2 \right) \Big|_a^1 \\ &= \frac{1}{4} \lim_{a \rightarrow 0^+} [(2 \ln 1 - 1) - (2a^2 \ln a - a^2)] = \frac{1}{4} \lim_{a \rightarrow 0^+} [a^2(1 - 2 \ln a) - 1] \end{aligned}$$

$$\lim_{a \rightarrow 0^+} [a^2(1 - 2 \ln a)] = \lim_{a \rightarrow 0^+} \frac{1 - 2 \ln a}{\frac{1}{a^2}} \stackrel{\infty}{=} \lim_{a \rightarrow 0^+} \frac{-\frac{2}{a}}{-\frac{2}{a^3}} = \lim_{a \rightarrow 0^+} \frac{2}{a} \cdot \frac{a^3}{2} = 0$$

L'Hôpital's Rule

$$\int_0^1 x \ln x dx = \frac{1}{4} \lim_{a \rightarrow 0^+} [a^2(1 - 2 \ln a) - 1] = \frac{1}{4} (0 - 1) = -\frac{1}{4} \in \mathbb{R}.$$

Therefore, the integral converges.

b) $x - e^{-x} = x - \frac{1}{e^x}$ is defined ($\forall x \gg 3$) and $x - e^{-x} < x$ ($\forall x \gg 3$) since $e^{-x} = \frac{1}{e^x} > 0$ for all $x \in \mathbb{R}$. Therefore $\frac{1}{x} < \frac{1}{x - e^{-x}}$ (for all $x \gg 3$). By the Direct Comparison Test $\int_3^\infty \frac{dx}{x - e^{-x}}$ diverges since $\int_3^\infty \frac{dx}{x}$ diverges by the p-test ($p=1$).

$$\text{OR } \frac{1}{x - e^{-x}} = \frac{e^x}{x e^x - 1} \not> \frac{e^x}{x e^x} = \frac{1}{x} \quad (\forall x \gg 3)$$

OR

$$\lim_{x \rightarrow +\infty} \frac{\frac{1}{x-e^{-x}}}{\frac{1}{x}} = \lim_{x \rightarrow +\infty} \frac{x}{x-e^{-x}} \stackrel{\frac{\infty}{\infty}}{\underset{\text{L'Hôpital's Rule}}{=}} \lim_{x \rightarrow +\infty} \frac{1}{1+e^{-x}} = 1 \Rightarrow$$

By the Limit Comparison Test, $\int_3^{\infty} \frac{dx}{x-e^{-x}}$ diverges

since $\int_3^{\infty} \frac{dx}{x}$ diverges (p-test, $p=1$).

c) $x \rightarrow 1^+ \Rightarrow \ln x \rightarrow 0^+ \Rightarrow \frac{1}{\ln x} \rightarrow +\infty$

$x \rightarrow 1^+ \Rightarrow x-1 \rightarrow 0^+ \Rightarrow \frac{1}{x-1} \rightarrow +\infty$

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) \stackrel{\infty-\infty}{=} \lim_{x \rightarrow 1^+} \frac{x-1-\ln x}{x \ln x - \ln x} \stackrel{\frac{0}{0}}{\underset{\text{L'Hôpital's Rule}}{=}} \lim_{x \rightarrow 1^+} \frac{1-\frac{1}{x}}{\ln x + 1 - \frac{1}{x}}$$

$$\stackrel{\frac{0}{0}}{\underset{\text{L'Hôpital's Rule}}{=}} \lim_{x \rightarrow 1^+} \frac{+\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2} //$$

L'Hôpital's Rule