

QUESTION 1

The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
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For the solution of this question please use only the front face and if necessary the back face of this page.

(7 pts) a) Find $\tan^{-1}(\tan \frac{3\pi}{4}) + \sin^{-1}(\sin \frac{\pi}{4}) + \sin(\cos^{-1} \frac{3}{5}) = ?$

(Hint: Consider the domains of the inverse trigonometric functions.)

(18 pts) b) Evaluate the following limits. (Do not use the L'Hopital's Rule)

(9 pts) (i) $\lim_{x \rightarrow 4} \ln \left(\frac{4x - x^2}{2 - \sqrt{x}} \right) = ?$

(9 pts) (ii) $\lim_{x \rightarrow +\infty} \frac{2x + \sin x}{x + 1} = ?$

a) $\tan^{-1}(\tan \frac{3\pi}{4}) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\sin^{-1}(\sin \frac{\pi}{4}) = \frac{\pi}{4}$

$\sin(\cos^{-1} \frac{3}{5}) = \frac{4}{5}$

$\tan^{-1}(\tan \frac{3\pi}{4}) + \sin^{-1}(\sin \frac{\pi}{4}) + \sin(\cos^{-1} \frac{3}{5}) = -\frac{\pi}{4} + \frac{\pi}{4} + \frac{4}{5} = \frac{4}{5}$

b) (i) $\lim_{x \rightarrow 4} \ln \left(\frac{4x - x^2}{2 - \sqrt{x}} \right) = \ln \left[\lim_{x \rightarrow 4} \left(\frac{4x - x^2}{2 - \sqrt{x}} \right) \right]$ since $y = \ln x$ is continuous

$= \ln \left[\lim_{x \rightarrow 4} \left(\frac{4x - x^2}{2 - \sqrt{x}} \right) \cdot \left(\frac{2 + \sqrt{x}}{2 + \sqrt{x}} \right) \right] = \ln \left[\lim_{x \rightarrow 4} \frac{x(4 - x)(2 + \sqrt{x})}{(4 - x)} \right]$

$= \ln [4 \cdot (2 + \sqrt{4})] = \ln 16$

(ii) $\lim_{x \rightarrow +\infty} \frac{2x + \sin x}{x + 1} = \lim_{x \rightarrow +\infty} \frac{x(2 + \frac{\sin x}{x})}{x(1 + \frac{1}{x})} = 2$

$\xrightarrow{x \rightarrow +\infty} 0$
 $\xrightarrow{x \rightarrow +\infty} 0$

QUESTION 2

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(12 pts) a) Is the function $f(x) = \begin{cases} \frac{\sin|x|}{x} & \text{if } -\frac{\pi}{2} < x < 0 \\ 1 & \text{if } x = 0 \\ x+1 & \text{if } 0 < x < 1 \\ 3 & \text{if } x = 1 \\ \sec(\frac{\pi x}{3}) & \text{if } 1 < x < \frac{4}{3} \end{cases}$ continuous on the open interval $(-\frac{\pi}{2}, \frac{4}{3})$?

If not, find the points of discontinuity of the function f and determine their types by evaluating the one-sided limits.

(13 pts) b) (i) State the Intermediate Value Theorem. (3 pts)

(ii) Using Intermediate Value Theorem, show that the equation $e^x + 2x = e$ has at least one solution (10 pts)a) Since $y = \frac{\sin|x|}{x}$ is cts on $(-\pi/2, 0)$, $y = x+1$ is cts on $(0, 1)$, $y = \sec(\frac{\pi x}{3})$ is cts on $(1, \frac{4}{3})$; we will consider the points $x=0, 1$

At $x=0$: $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin|x|}{x} = \lim_{x \rightarrow 0^-} \frac{\sin(-x)}{x} = \lim_{x \rightarrow 0^-} -\frac{\sin x}{x} = -1$

& $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x+1) = 1 = f(0) \Rightarrow f$ is not cts at $x=0$, since $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$.

TYPE: "Jump" discontinuity.

At $x=1$: $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x+1) = 2$ & $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sec(\frac{\pi x}{3}) = 2$

$\Rightarrow \lim_{x \rightarrow 1} f(x) = 2$. Since $f(1) = 3 \neq \lim_{x \rightarrow 1} f(x)$, f is not cts at $x=1$

TYPE: "Removable" discontinuity.

b) (i) The Intermediate Value Theorem: A function

$y=f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

OR

Let f be a continuous function on the closed interval $[a,b]$. If y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c \in [a,b]$.

(ii) Let $f(x) = e^x + 2x - e$. Then f is continuous everywhere.
 $f(0) = 1 - e < 0$
 $f(1) = e + 2 - e = 2 > 0$ } By (a corollary to) the Intermediate Value Theorem f has at least one zero in $(0,1)$.

Therefore, $e^x + 2x - e = 0$ has at least one solution.

QUESTION 3

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(10 pts) a) Let f be a function satisfying the equation $f(x+h) = f(x) + f(h) + x^2h + xh^2$ for all $x, h \in \mathbb{R}$.Suppose that $\lim_{h \rightarrow 0} \frac{f(h)}{h} = 1$.(i) Show that $f(0) = 0$.(ii) By using the limit definition of the derivative find $f'(x)$.(5 pts) b) Find $\frac{dy}{dx}$ where $y = (\sec x + \tan x)^2 + (x^2 + 1)^{5/2}$.(10 pts) c) Find the equation of the tangent line to the parameterized curve $x(t) = \sin t$, $y(t) = \sqrt{t+1}$ at $t = 0$.

$$\text{a) (i)} \quad \underset{\uparrow x}{f(0+0)} = \underset{\uparrow h}{f(0)} + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 = 0 \Rightarrow f(0) = 2f(0) \Rightarrow \underline{f(0) = 0}$$

$$\begin{aligned} \text{(ii)} \quad \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{f(x) + f(h) + x^2h + xh^2 - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h) + h(x^2 + xh)}{h} = \lim_{h \rightarrow 0} \left[\frac{f(h)}{h} + (x^2 + xh) \right] = \underline{1 + x^2} \\ &\quad \text{since } \lim_{h \rightarrow 0} \frac{f(h)}{h} = 1. \end{aligned}$$

$$\text{b)} \quad y = (\sec x + \tan x)^2 + (x^2 + 1)^{5/2}$$

$$\begin{aligned} \frac{dy}{dx} &= 2(\sec x + \tan x) \cdot (\sec x + \tan x)' + \frac{5}{2}(x^2 + 1)^{3/2} \cdot (x^2 + 1)' \\ &= 2(\sec x + \tan x) \cdot (\sec x \tan x + \sec^2 x) + \frac{5}{2}(x^2 + 1)^{3/2} \cdot 2x \end{aligned}$$

$$\text{c)} \quad \left. \begin{aligned} x(t) &= \sin t \Rightarrow \frac{dx}{dt} = \cos t \\ y(t) &= \sqrt{t+1} \Rightarrow \frac{dy}{dt} = \frac{1}{2\sqrt{t+1}} \end{aligned} \right\} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2\sqrt{t+1}}}{\cos t} = \frac{1}{2\sqrt{t+1} \cdot \cos t}$$

$$\eta_{\text{tangent}} = \left. \frac{dy}{dx} \right|_{t=0} = \underline{\frac{1}{2}}$$

$$t=0 \Rightarrow x=0, y=1$$

$$P(0,1), m_{\text{tangent}} = 1/2 \Rightarrow y-1 = \frac{x}{2} \Rightarrow \boxed{y = \frac{x}{2} + 1} \text{ is the tangent line at } P$$

QUESTION 4

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(25 pts) Let $f(x) = \frac{x^2 + 1}{(x-1)^2}$.

(i) Find the domain and, if any, the intercept(s) of $f(x)$. $D_f: (-\infty, +1) \cup (+1, +\infty)$

$x=0 \Rightarrow y=1 \Rightarrow (0,1)$ is the y -intercept of f .

Since $x^2+1 > 0$ for any $x \in D_f$, there is no x -intercept for f .

(ii) If any, find the horizontal asymptote(s) of $f(x)$.

$\lim_{x \rightarrow \pm\infty} \frac{x^2+1}{(x-1)^2} = 1 \Rightarrow y=1$ is the horizontal asymptote of f .

(iii) If any, find the vertical asymptote(s) of $f(x)$.

$\lim_{x \rightarrow 1^-} \frac{x^2+1}{(x-1)^2} = +\infty$, $\lim_{x \rightarrow 1^+} \frac{x^2+1}{(x-1)^2} = +\infty \Rightarrow x=1$ is the vertical asymptote of f .

(iv) If any, find the oblique asymptote(s) of $f(x)$.

There is no oblique asymptote for f .

(v) Find the intervals on which the function is increasing and decreasing, and identify the function's local extreme value(s), if any, saying where they are taken on.

$$f'(x) = \frac{2x(x-1)^2 - (x^2+1)2(x-1)}{(x-1)^4} = \frac{2(x-1)[x(x-1) - (x^2+1)]}{(x-1)^4} = \frac{2(x^2 - x - x^2 - 1)}{(x-1)^3} = \frac{-2(x+1)}{(x-1)^3}$$

$$\Rightarrow f'(x) = -\frac{2(x+1)}{(x-1)^3} = 0 \Rightarrow x = -1$$

$x \neq 1$ Critical point

x	$f'(x)$	Sign
$x < -1$	$f' < 0$	-
$x = -1$	$f' = 0$	0
$-1 < x < 1$	$f' > 0$	+
$x = 1$	f' undefined	
$x > 1$	$f' < 0$	-

- f is increasing on $(-1, 1)$
- f is decreasing on $(-\infty, -1) \cup (1, +\infty)$
- f has a local minimum at $x = -1$

(vi) Identify the concavity and, if any, find the point(s) of inflection.

$$f''(x) = \frac{-2(x-1)^3 + 2(x+1)3(x-1)^2}{(x-1)^6} = \frac{2(x-1)^2[3(x+1) - (x-1)]}{(x-1)^6} = \frac{4x+8}{(x-1)^4} = \frac{4(x+2)}{(x-1)^4}$$

x	$f''(x)$	Sign
$x < -2$	$f'' < 0$	-
$x = -2$	$f'' = 0$	0
$-2 < x < 1$	$f'' > 0$	+
$x = 1$	f'' undefined	
$x > 1$	$f'' > 0$	+

- f is concave down on $(-\infty, -2)$.
- f is concave up on $(-2, \infty)$ and $x = -2$ is an inflection point.

(vii) By using all obtained above, fill in the following table and graph the curve of $y = f(x)$ on the back side of the paper.

	$-\infty$	-2	-1	1	$+\infty$
f'	-	-	0	+	-
f''	-	0	+	+	+

$y=1$ (horizontal asymptote)
 $x=1$ (vertical asymptote)
 Graph shows a curve with a local minimum at $x=-1$ and an inflection point at $x=-2$, approaching $y=1$ as $x \rightarrow \pm\infty$ and $x \rightarrow 1$.

